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Trees with Given Independence Number Maximizing the A_α-Spectral Radius

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Abstract. Spectral graph theory is a widely studied and applied subject in combinatorial mathematics, computer science and social science. Nikiforov (2017) defined a convex linear combination for the graph G, denoted by $A_\alpha(G)(\alpha \in [0,1])$. This concept can be regarded as a common generalization of adjacency matrix and unsigned Laplacian matrix. We mainly study the A_α-spectral extreme problem for graphs, which is a generalization of Brualdi and Solheid's problem on A_α-matrices. Let $T_{n,\gamma}$ be the set of all trees with order n and independence number γ. By graph transformations we determine the graphs with maximal A_α-spectral radius among $T_{n,\gamma}$ for $\lceil n/2 \rceil \le \gamma \le n-1$. Therefore, we extend the results of Ji and Lu (2016) from spectral radius to A_α-spectral radius.

Keywords: A_α-spectral radius, independence number, tree

1. Introduction

This paper only considers simple and undirected graphs. Let G be a graph. $A(G)$ and $D(G)$ denote the adjacency matrix and diagonal matrix of G, respectively. The largest eigenvalue of $A(G)$, i.e. the spectral radius of G is denoted by $\rho(G)$. Let $Q(G) = D(G) + A(G)$ denote the signless Laplacian matrix of G. We use $q(G)$, i.e. the largest eigenvalue of $Q(G)$, to denote the signless Laplacian spectral radius of G. When G is connected, $A(G)$ is irreducible. Based on the study of $A(G)$ and $Q(G)$, Nikiforov[1] proposed a new class of graph matrices $A_\alpha(G)$, which defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \alpha \in [0,1]$. Clearly, $A_0(G)(= A(G))$ and $A_{1/2}(G)(= Q(G))$ are the the adjacency matrix and signless Laplacain matrix of G, respectively. Suppose that $\alpha = 1$ then $A_1(G) = D(G)$. We use $\rho(A_\alpha(G))$, i.e. the largest eigenvalue of $A_\alpha(G)$, to denote the A_α-spectral radius of G. According to the Perron-Frobenius Theorem, we know that the spectral radius is simple and has a unique positive eigenvector. Thus, the eigenvector of $\rho(A_\alpha(G))$ is also called the A_α-Perron vector of G.

The spectral theory of graph generally encodes the structure of graph into matrix, and then tracks the relationship between the properties and eigenvalues of graph. It is widely used in combinatorial mathematics, computer science, social science and other fields. Motivated by the general problem of Brualdi and Solheid [2], the problem concerning graphs with the extremal spectral radius of restricted graph is therefore of interest, one can refer to [3-12] and reference herein. Recently, many authors have studied the relationship between spectral radius and independence number. For example, among the graphs with independence number $\gamma \in \{1, 2, \lceil n/2 \rceil, \lceil n/2 \rceil + 1, n-3, n-2, n-1\}$ the graphs with minimal spectral radius was determined by Xu et al. [4] ; Feng et al. [9] studied the spectral radii of unicyclic graphs with given independence number; Ji et al. [12] determined the graphs with maximal spectral radius among all the trees with given independence number, etc. In addition, some problems about A_α-spectral radius have aroused the interest of the authors, such as, Rojo [13] obtained a general result on the A_α-spectrum of copies of a rooted graph; furthermore, Rojo [14] proved that $(n-d+2)\alpha - 1$ is an eigenvalue of $A_\alpha(B)$ with multiplicity $n-d-1$, for a bug B of order n and diameter d; Lin et al.[15] characterized the extremal graphs with maximal A_α-spectral radius for fixed order and cut vertices (or fixed order and matching number); Nikiforov et al. [16]

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studied the distribution of the entries of Perron vectors along pendent paths in graphs for A_α-spectral radius; Li et al.[17] characterized all extremal trees and extremal unicyclic graphs with the maximum A_α-spectral radius for prescribed degree sequence, respectively. For further results, one can refer to [18-20].

Let $\gamma(G)$ be the independence number of G. A tree is a connected acyclic graph. The main purpose of this paper is to study the A_α-spectral radius of graph with fixed order and independence number. The extremal graphs with maximal A_α-spectral radius among all the trees with order n and independence number γ ($\lceil n/2\rceil \le \gamma \le n-1$)r for $\alpha \in [0,1)$ are determined.

2. Main Results

Let $G=(V(G),E(G))$ be a graph. For $v\in V(G)$, let $N_G(v)$ denote the adjacent vertex set of v and $d_G(v)=|N_G(v)|$. Let $u,v\in V(G)$. $d_G(u,v)$ denotes the distance between u and v in G. Suppose that $H\subseteq G$. Then for $v\in V(H)$, we define $N_H(v)=N_G(v)\bigcap V(H)$ and $d_H(v)=|N_H(v)|$. $G[S]$ denotes the subgraph induced by $S\subseteq V(G)$. We call the vertex of degree one as a leaf, and call the edge incident with it as a pendant edge. A support vertex is one that has leaves as its neighbors. Suppose $e=(x,y)\in E(G)$. $G-(x,y)$ denotes the graph obtained from G by deleting the edge e. Suppose $x,y\in V(G)$ and $(x,y)\notin E(G)$. $G+(x,y)$ denotes the graph obtained from G by adding an edge (x,y).

Theorem 2.1 Let $G=(V(G),E(G))$ be a connected graph. $\rho(A_\alpha(G))$ denotes the A_α-spectral radius of G. For u, $v\in V(G)$, suppose v_1, v_2, $\cdots$, v_s $\in N_G(v)\setminus N_G(u)$ $(1\le s\le d_G(v))$ and $x=(x_1,x_2,\cdots,x_n)^T$ is the A_α-Perron vector of G, where x_i corresponds to the vertex v_i $(1\le i\le n)$. Let $G^*-(v,v_i)+(u,v_i)$, $(1\le i\le s)$. Then $\rho(A_\alpha(G))<\rho(A_\alpha(G^*))$ if $x_u\ge x_v$.

Proof. Since $x^T(A_\alpha(G^*)-A_\alpha(G))x=2(1-\alpha)\sum_{i=1}^{s}x_i(x_u-x_v)+s\alpha(x_u^2-x_v^2)\ge 0.$ Then

$$\rho(A_\alpha(G^*))=\max_{\|y\|=1}y^TA_\alpha(G^*)y\ge x^TA_\alpha(G^*)x\ge x^TA_\alpha(G)x=\rho(A_\alpha(G)) \tag{1}$$

Suppose that $\rho(A_\alpha(G^*))=\rho(A_\alpha(G))$. Then we have $\rho(A_\alpha(G^*))=x^T(A_\alpha(G^*))x.$
From $(A_\alpha(G^*))x=\rho(A_\alpha(G^*))x$, we get

$$\rho(A_\alpha(G^*))x_v=((A_\alpha(G^*))x)_v=\sum_{v_i\in N_{G^*}(v)}x_i+(d_v-s)x_v. \tag{2}$$

Also, from $(A_\alpha(G))x=\rho(A_\alpha(G))x$ we have

$$\rho(A_\alpha(G))x_v=((A_\alpha(G))x)_v=\sum_{v_i\in N_G(v)}x_i+d_vx_v=\sum_{v_i\in N_{G^*}(v)}x_i+\sum_{i=1}^{s}x_i+d_vx_v. \tag{3}$$

Since $x=(x_1,x_2,\cdots,x_n)^T$ is the A_α-Perron vector of $A_\alpha(G)$, $x_i>0$ $(1\le i\le n)$. Thus $\sum_{i=1}^{s}x_i\ge 0$. By (2) and (3), we know that $\rho(A_\alpha(G^*))x_v<\rho(A_\alpha(G))x_v$. Then $\rho(A_\alpha(G^*))<\rho(A_\alpha(G))$, a contradiction. The proof is complete.

Lemma 2.1([12]) For any tree T, let $L(T)$ be the set of the leaves of T. Then there exists a maximum independent set $S(T)$ of T such that $L(T)\subseteq S(T)$.

Definition 2.1([12]) Let S_n be a star with n vertices. $T_{n,\gamma}$ denotes the set of trees of order n with independence number γ ($\lceil n/2\rceil\le\gamma\le n-1$). Suppose $\gamma=n-t$ ($1\le t\le\lfloor n/2\rfloor$). $S^1_{n,n-t}$ denotes a set of trees of order n obtained from S_t ($V(S_t)=\{v_0,v_1,\cdots,v_{t-1}\}$ and $d_{S_t}(v_0)=t-1$) by attaching at least one pendant edge to each vertex of $\{v_0,v_1,\cdots,v_{t-1}\}$ so that the total number of the leaves in $S^1_{n,n-t}$ is $n-t$; $S^2_{n,n-t}$ denotes a set of order n obtained from S_{t+1} ($V(S_{t+1})=\{v_0,v_1,\cdots,v_t\}$ and $d_{S_{t+1}}(v_0)=t$) by attaching at least one pendant edge to each vertex of $\{v_0,v_1,\cdots,v_t\}$ so that the total number of the leaves in $S^2_{n,n-t}$ is $n-t-1$($t\ne n/2$); $S^*_{n,n-t}$ denotes the tree of order n obtained from S_t ($V(S_t)=\{v_0,v_1,\cdots,v_{t-1}\}$ and $d_{S_t}(v_0)=t-1$) by attaching one pendant edge to each vertex of $\{v_0,v_1,\cdots,v_{t-1}\}$ and attaching $n-2t+1$ vertices to v_0.

Remark. $T_{n,n-1} = S_n$ and $S^*_{n,n-t} \in S^1_{n,n-t} \subseteq T_{n,n-t}$.

Lemma 2.2 Suppose $T \in T_{n,n-t}$ such that the A_α-spectral radius of T is as large as possible. Then $T \in S^1_{n,n-t}$.

Proof. Let $x = (x_1, x_2, \cdots, x_n)^T$ be the A_α-Perron vector of T, where x_i corresponds to the vertex $v_i (1 \le i \le n)$. By Definition 2.1, suppose $T \in T_{n,n-t} \setminus (S^1_{n,n-t} \cup S^2_{n,n-t})$. We know that the number of the support vertices of $T[V(T) \setminus L(T)]$ is at least 2. Let $v_1, v_2 \in T[V(T) \setminus L(T)]$, such that $d_{T[V(T)\setminus L(T)]}(v_1, v_2)$ is as large as possible. Then T is shown as in Figure 1 (See the proof of Theorem 3 in [12]), where H denotes the component of $T - (v_1, v_3) - (v_2, v_4)$ containing the vertices v_3, v_4 (possibly $v_3 = v_4$) if $(v_1, v_2) \notin E(T)$; if $(v_1, v_2) \in E(T)$ then $H = \varnothing$.

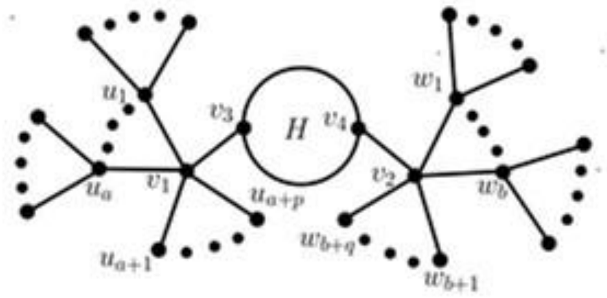

Fig. 1: $T \in T_{n,n-t} \setminus (S^1_{n,n-t} \cup S^2_{n,n-t})$.

By Lemma 2.1, there exists a maximum independent set $S(T)$ in T such that $L(T) \subseteq S(T)$. By Theorem 2.1, we construct T_1 from T such that $T_1 = T - \sum_{i=1}^{a}(v_1, u_i) + \sum_{i=1}^{a}(v_2, u_i)$ if $x_{v_1} \le x_{v_2}$; otherwise

$$T_1 = T - \sum_{i=1}^{b}(v_2, w_i) + \sum_{i=1}^{b}(v_1, w_i).$$

Let $S(T_1)$ be a maximum independent set of T_1 ($L(T_1) \subseteq S(T_1)$). Refer to the Fig. 1, we distinguish with the following three cases.

Case 1 $p, q > 0$. Then $\gamma(T) = |\ S(T)| = \sum_{i=1}^{a}(d_T(u_i) - 1) + p + \sum_{i=1}^{b}(d_T(w_i) - 1) + q + \gamma(H)$ and

$$\gamma(T_1) = |\ S(T_1)| = \sum_{i=1}^{a}(d_{T_1}(u_i) - 1) + p + \sum_{i=1}^{b}(d_{T_1}(w_i) - 1) + q + \gamma(H).$$

Thus $\gamma(T_1) = \gamma(T)$. Then $T_1 \in T_{n,n-t}$. By Theorem 2.1, $\rho(A_\alpha(T)) < \rho(A_\alpha(T_1))$, a contradiction.

Case 2 Only one of p and q equals to zero. Suppose, without loss of generality, that $p = 0$ but $q > 0$. Then $\gamma(T) = |\ S(T)| = \sum_{i=1}^{a}(d_T(u_i) - 1) + \sum_{i=1}^{b}(d_T(w_i) - 1) + q + \gamma(T[V(H) \cup \{v_1\}])$ and

$$\gamma(T_1) = |\ S(T_1)| = \sum_{i=1}^{a}(d_{T_1}(u_i) - 1) + \sum_{i=1}^{b}(d_{T_1}(w_i) - 1) + q + \gamma(T_1[V(H) \cup \{v_1\}]).$$

Hence $\gamma(T_1) = \gamma(T)$. Then $T_1 \in T_{n,n-t}$. By Theorem 2.1 we have $\rho(A_\alpha(T)) < \rho(A_\alpha(T_1))$, a contradiction.

Case 3 $p = q = 0$. Then $\gamma(T) = |\ S(T)| = \sum_{i=1}^{a}(d_T(u_i) - 1) + \sum_{i=1}^{b}(d_T(w_i) - 1) + \gamma(T[V(H) \cup \{v_1, v_2\}])$ and

$$\gamma(T_1) = |\ S(T_1)| = \sum_{i=1}^{a}(d_T(u_i) - 1) + \sum_{i=1}^{b}(d_T(w_i) - 1) + \gamma(T[V(H) \cup \{v_1, v_2\}]).$$

Hence $\gamma(T_1) = \gamma(T)$. Then $T_1 \in T_{n,n-t}$. By Theorem 2.1 we get $\rho(A_\alpha(T)) < \rho(A_\alpha(T_1))$, a contradiction.

From Cases 1-3 we know that $T \in T_{n,n-t} \setminus (S^1_{n,n-t} \cup S^2_{n,n-t})$. In the following we show that $T \notin S^2_{n,n-t}$.

Suppose $T \in S^2_{n,n-e}$. Let $N(v_0) = \{v_1, v_2, \cdots, v_t\}$ and $N(v_1) = \{v_0, u_1, u_2, \cdots, u_s\}$. We construct T_2 from T such that $T_2 = T - \sum_{i=1}^{s}(v_1, u_i) + \sum_{i=1}^{s}(v_0, u_i)$ if $x_{v_1} \le x_{v_0}$; otherwise $T_2 = T - \sum_{k=1}^{t}(v_0, v_k) + \sum_{k=1}^{t}(v_1, v_k)$.

Then in either case, $T_2 \in S^1_{n,n-t}$. By Theorem 2.1, we know that $\rho(A_\alpha(T)) < \rho(A_\alpha(T_2))$, a contradiction. Therefore $T \in S^1_{n,n-t}$. The proof is complete.

Theorem 2.2 Let $T \in T_{n,n-t}$ with $t \geq 2$. Then $\rho(A_\alpha(T)) \leq \rho(A_\alpha(S^*_{n,n-t}))$ and equality holds if and only if $T = S^*_{n,n-t}$.

Proof. Suppose $T \in T_{n,n-t}$ such that the A_α-spectral radius of T is as large as possible. Let $x = (x_1, x_2, \cdots, x_n)^T$ be a A_α-Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. By Lemma 2.2, $T \in S^1_{n,n-t}$. Since $S^*_{n,n-t} \in S^1_{n,n-t} \subseteq T_{n,n-t}$.

Suppose $T \notin S^*_{n,n-t}$. Then, wlog., we assume that $d_{v_1} \geq 3$. Let $N(v_1) = \{v_0, u_1, u_2, \cdots, u_s\}$ and $N(v_0) = \{v_1, v_2, \cdots, v_{t-1}, w_1, w_2, \cdots, w_p\}$, where $d(w_1) = \cdots = d(w_p) = 1$. Hence we get $s \geq 2$ and $p \geq 1$.

If $t = 2$ then $p \geq 2$. Assume, wlog., that $x_{v_1} \leq x_{v_0}$. Let $T^{'} = T - \sum_{i=2}^{s} v_1 u_i + \sum_{i=2}^{s} v_0 u_i$. Then $T^{'} \in S^*_{n,n-2}$ and $\rho(A_\alpha(T)) < \rho(A_\alpha(S^*_{n,n-2}))$. By Theorem 2.1, a contradiction. Thus we will assume that $t \geq 3$.

We construct T_3 from T such that $T_3 = T - \sum_{i=2}^{s}(v_1, u_i) + \sum_{i=2}^{s}(v_0, u_i)$ if $x_{v_1} \leq x_{v_0}$; otherwise, if $p \geq 2$ then $T_3 = T - \sum_{k=2}^{t-1}(v_0, v_k) + \sum_{j=2}^{p}(v_0, w_j) + \sum_{k=2}^{t-1}(v_1, v_k) + \sum_{j=2}^{p}(v_1, w_j)$; and if $p = 1$ then $T_3 = T - \sum_{k=2}^{t-1} v_0 v_k + \sum_{k=2}^{t-1} v_1 v_k$. From Theorem 2.1, in either above cases we have $\rho(A_\alpha(T)) < \rho(A_\alpha(T_3))$. Also, we note that $\gamma(T_3) = \gamma(T) = n - t$, we get a contradiction.

Remark. When $\alpha = 0, 1/2$. Theorem 2.2 implies that the results of [12](Theorem 3 and Theorem 5) .

3. Conclusions

In fact, spectral graph theory is related to complex networks and is widely used in computer science. This paper mainly focuses on the problem of A_α-spectral radius problem, which is a generalization of the general problem of Brualdi and Solheid [2] on A_α-matrix. We study the A_α-spectral radius of graph with fixed order and independence number. By graph transformations the graphs with maximal A_α-spectral radius among $T_{n,\gamma}$ for $\lceil n/2 \rceil \leq \gamma \leq n-1$ are determined. These results generalize the Theorem 3 and Theorem 5 in [12] (Ji and Lu, Linear Algebra Appl. 488(2016) 102-108).

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